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Abstract
We construct a family of explicit rotational solutions to the nonlinear governing equations for water waves, describing edge waves propagating over a plane-sloping beach. A detailed analysis of the edge wave dynamics and of the run-up pattern is made possible by the use of the Lagrangian approach to the water motion.

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1. Introduction

Standing on a gently sloping straight beach, it is a matter of observation that various waveforms propagate on the surface of the sea. Among these we find edge waves—water waves that progress along the shoreline. These waves, often difficult to visualize (this has, probably, prevented the regarding of this waveform as important for a long time), are coastal trapped, i.e. their amplitude is maximal at the shoreline and decays rapidly offshore. They produce on the beach beautiful run-up patterns (highest points reached by a wave on the beach). Although propagation is along the straight shoreline and the waveform is sinusoidal in the longshore, these waves are not one dimensional [1].

While they were originally considered to be a curiosity [2], edge waves are now recognized to play a significant role in nearshore hydrodynamics. For shallow beaches empirical evidence shows that incident storm waves lose most of their energy through wave-breaking by the time they reach the shore. After breaking offshore, as the waves progress to shallower water, their height decreases reaching its least value at the shoreline. Since storms often result in pronounced shoreline erosion, the surf-zone water processes with the onset of a storm are dominated by wave conditions other than the incident waves—role attributed by oceanographers to the edge waves [3]. There are other instances where edge waves are of significance. For example, processed data from the water waves created by an earthquake occurring in April 1992 in the ocean floor near the Californian coast show that two distinct wave packets (both directly generated in the nearshore by the vertical motion of the ocean bottom) were recorded at a coastal station about 150 km from the epicentre. At first, less than an hour after the occurrence of the earthquake, a relatively fast-moving swell with an amplitude around 15 cm struck the offshore. About two hours after the swell had subsided, relatively slow-moving edge waves with amplitudes around 50 cm [4] arrived. Measurements performed on this occasion confirm the rapid decay of the amplitude of the edge waves: at an offshore distance of 12 km the amplitude is reduced to 10% of its maximal value (attained at the shoreline). Let us also mention that it has been observed [5] that hurricanes travelling approximately parallel to a nearby coastline sometimes give rise to edge waves. Interestingly, edge waves can in fact be generated directly in a laboratory wave-tank [6].

The edge wave phenomenon has been extensively studied and discussed in the mathematical literature within the framework of linear theory. Due to the small displacements associated with these waves, the governing equations for water waves or the shallow-water equations are linearized (Minzoni and Whitham [7] showed that both approximations are equally consistent) and this simplification permits a thorough analysis. We refer to Ehrenmark [8] for an up-to-date survey.

Despite the fact that the linearizing approximation lacks rigorous mathematical justification, it has been used with considerable success as a large variety of theoretical studies are confirmed in experimental contexts. The investigation of nonlinear edge waves can be seen as a natural extension to the linear theory. Whitham [9] showed the existence of irrotational weakly nonlinear edge waves that propagate parallel to the shore using a formal Fourier series expansion for the full water-wave theory. A study of properties of nonlinear progressive edge waves based on the fact that the evolution is described by the nonlinear Schrödinger equation was carried out by Yeh [6]. This paper describes an alternative approach; the main impetus for the results reported here comes from the belief that the need for a more rigorous theory remains thoroughly justified. A quest for an explicit edge-wave solution for the governing equations appears to be of interest since the structure of the edge waves obscures their visual observation. Moreover, a solution in closed mathematical form provides a background against which certain features which have been observed (and predicted) can be checked, and it may also highlight the underlying physical processes more readily than a computationally intensive approach. It turns out that the deep water wave solution discovered by Gerstner [10] can be adapted to construct edge waves propagating along a plane-sloping beach. This possibility was pointed out by Yih [11] but the treatment therein, in essence followed also by Mollo-Christensen [12], provides only an implicit form for the free water surface. We present a procedure by which exact edge-wave solutions to the full water-wave equations are obtained. The closed form of the solution in Lagrangian (material) coordinates permits us to provide clear illustrations of the structure of these edge waves. From an examination of the solution we also obtain the run-up pattern, an attractive feature being the occurrence of cusps. The obtained run-up shapes are confirmed in both field and laboratory evidence [3]. Thus, we establish with rigour the existence of rotational nonlinear edge waves, unravelling the detailed structure of the wave pattern.

2. The edge wave

We take a plane beach and adopt a coordinate system as shown below, with shoreline being the x-axis and still sea in the region

$$R = \{(x, y, z) : x \in \mathbb{R},\ y \leqslant b_0,\ 0 \leqslant z \leqslant (b_0 - y) \tan\alpha\}$$

for some $b_0 \leqslant 0$; here $\alpha \in (0, \frac{\pi}{2})$ defines the uniform slope.

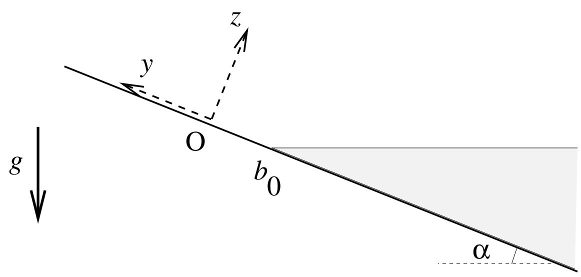

Figure 1. Cross section of the still sea.

Let $u = (u_1, u_2, u_3)$ be the velocity field and let us recall (e.g. [13]) the governing equations for the propagation of gravity water waves when ignoring viscous effects. Homogeneity (constant density ρ) is a good approximation for water (see the numerical data in Lighthill [1]) so that we have the equation of mass conservation in the form

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} = 0. \tag{1}$$

The equation of motion is Euler's equation

$$\begin{cases} \dfrac{\mathrm{D}u_1}{\mathrm{D}t} = -\dfrac{1}{\rho}\,\dfrac{\partial P}{\partial x} \\[2ex] \dfrac{\mathrm{D}u_2}{\mathrm{D}t} = -\dfrac{1}{\rho}\,\dfrac{\partial P}{\partial y} - g\sin\alpha \\[2ex] \dfrac{\mathrm{D}u_3}{\mathrm{D}t} = -\dfrac{1}{\rho}\,\dfrac{\partial P}{\partial z} - g\cos\alpha \end{cases} \tag{2}$$

where $P(t,\ x,\ y,\ z)$ denotes the pressure, g is the gravitational acceleration constant and $\mathrm{D}/\mathrm{D}t$ is the material time derivative,

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + u_1\,\frac{\partial f}{\partial x} + u_2\,\frac{\partial f}{\partial y} + u_3\,\frac{\partial f}{\partial z}$$

expressing the rate of change of the quantity f associated with the same fluid particle as it moves about. The boundary conditions which select the water-wave problem from all other possible solutions of the equations (1) and (2) are [14]:

(i) the dynamic boundary condition $P = P_0$ at the free surface, where P_0 is the constant atmospheric pressure, decouples the motion of the air from that of the water;

(ii) the kinematic boundary condition at the free surface expresses the fact that the same particles always form the free water surface;

(iii) the kinematic boundary condition at the bottom, requiring the normal velocity component at the bed to be zero so that it is impossible for water to penetrate.

The general description of the propagation of a water wave is encompassed by the equations (1) and (2), and the three boundary conditions (i)–(iii), a distinctive feature being that the free surface is not known and must be determined as part of the solution.

We adopt the Lagrangian point of view by following the evolution of individual water particles. We suppose that the position of a particle at time t is given by

$$\begin{cases} x = a - \dfrac{1}{k}\,\mathrm{e}^{k(b-c)} \sin\left(ka + \sqrt{gk\,\sin\alpha}\,t\right) \\ y = b - c + \dfrac{1}{k}\,\mathrm{e}^{k(b-c)} \cos\left(ka + \sqrt{gk\,\sin\alpha}\,t\right) \\ z = c + c\,\tan\alpha - \dfrac{\tan\alpha}{2k}\,\mathrm{e}^{2kb_0}\left(1 - \mathrm{e}^{-2kc\,(1+\cot\alpha)}\right) \end{cases} \tag{3}$$

where $k > 0$ is fixed. It should be pointed out that the quantities a, b and c do not stand for the initial coordinates of a particle, but are simply labelling variables serving to identify a particle. We may think of them as parameters which fix the position of a particular particle before the passage of the wave (in still water), despite the fact that the wave is not developing from the still state, otherwise, the flow would be irrotational in view of Helmholtz's theorem [14] but its vorticity is nonzero (see the last section). Let us explain the origin of (3). Gerstner [10] gave the only known nontrivial explicit solution to the full water-wave equations, showing that the two-dimensional particle motion ($a \in \mathbb{R},\ b \leqslant b_0 \leqslant 0,\ k > 0$)

$$t \mapsto \left(a + \frac{1}{k}\,\mathrm{e}^{kb}\sin(ka + \sqrt{gk}\,t),\ b - \frac{1}{k}\,\mathrm{e}^{kb}\cos(ka + \sqrt{gk}\,t)\right)$$

represents waves of finite amplitude in water of infinite depth. This suggests that it might be possible to construct an edge wave using an approach similar to that for a Gerstner wave field. While the theoretical correctness of this conclusion was established by Yih (1966) and Mollo-Christensen [12], the outcome in both treatments was an implicit form on the water's free surface which makes the obtained waveform graphically and computationally inaccessible. The closed form (3) provides the full details of the edge-wave motion without considerable labour.

Our aim is to prove that the motion (3) is dynamically possible and that we can associate with it an expression for the hydrodynamical pressure P such that the governing equations and boundary conditions are all satisfied. The resulting free surface of the water will be the edge wave we are looking for.

The map (3) is a diffeomorphism from the still water region R to the water region, bounded below by the rigid bed $\{z = 0\}$ and above by the free water surface, parametrized by

$$\begin{cases} x = a - \dfrac{1}{k}\,\mathrm{e}^{kb(1+\tan\alpha)-kb_0\tan\alpha} \sin\left(ka + \sqrt{gk\,\sin\alpha}\,t\right) \\ y = b(1+\tan\alpha) - b_0\tan\alpha + \dfrac{1}{k}\,\mathrm{e}^{kb(1+\tan\alpha)-kb_0\tan\alpha} \cos\left(ka + \sqrt{gk\,\sin\alpha}\,t\right) \\ z = (b_0 - b)\,(1+\tan\alpha)\tan\alpha - \dfrac{\tan\alpha}{2k}\,\mathrm{e}^{2kb_0}\left(1 - \mathrm{e}^{2k(b-b_0)\,(1+\tan\alpha)}\right) \end{cases} \tag{4}$$

with $a \in \mathbb{R}$, $b \leqslant b_0$ and $t \geqslant 0$. Indeed, observing that $(a, b, c) \mapsto (a, b - c, c)$ defines a diffeomorphism of $\mathbb{R}^3$, it is enough to show that the map

$$\begin{pmatrix} a \\ b' \\ c \end{pmatrix} \mapsto \begin{pmatrix} a - \dfrac{1}{k}\,\mathrm{e}^{kb'} \sin\left(ka + \sqrt{gk\,\sin\alpha}\,t\right) \\ b' + \dfrac{1}{k}\,\mathrm{e}^{kb'} \cos\left(ka + \sqrt{gk\,\sin\alpha}\,t\right) \\ c + c\,\tan\alpha - \dfrac{\tan\alpha}{2k}\,\mathrm{e}^{2kb_0}\left(1 - \mathrm{e}^{-2kc\,(1+\cot\alpha)}\right) \end{pmatrix}$$

is a diffeomorphism on $\mathbb{R} \times \mathbb{R}_- \times \mathbb{R}_+$. To see this, observe that the third coordinate depends only on c, being an increasing function $f(c)$ with $f(0) = 0$ and $\lim_{c\to\infty} f(c) = \infty$. Therefore, we may slice $\mathbb{R} \times \mathbb{R}_- \times \mathbb{R}_+$ by planes parallel to the plane $c = 0$ and in each such plane the approach used in the case of a Gerstner wave field [15] is applicable. This proves that (3) is a diffeomorphism and it is easy to identify the boundary of the image of the region R under it.

The Lagrangian form of the equation of continuity (that is, the volume-preserving property of the flow) is fulfilled since the value of the Jacobian of the map (3) is independent of time. This, together with the previously proved fact that (3) defines at any fixed time a diffeomorphism, shows that the motion described by (3) is dynamically possible. To complete the proof that (3) describes the water motion induced by a gravity wave, we have to check Euler's equation (2) and the boundary conditions (i)–(iii) for a suitably defined value of the hydrodynamical pressure.

The acceleration of a particular water particle is

$$\frac{\mathrm{D}u}{\mathrm{D}t} = \left(g \sin\alpha\, \mathrm{e}^{k(b-c)} \sin(ka + \sqrt{gk \sin\alpha}\, t),\ -g \sin\alpha\, \mathrm{e}^{k(b-c)} \cos(ka + \sqrt{gk \sin\alpha}\, t)\,,\ 0\right)$$

so that the equation of motion (2) is

$$\begin{cases} \dfrac{\partial P}{\partial x} = -\rho g \sin\alpha\, \mathrm{e}^{k(b-c)} \sin(ka + \sqrt{gk \sin\alpha}\, t) \\[2mm] \dfrac{\partial P}{\partial y} = \rho g \sin\alpha\, \mathrm{e}^{k(b-c)} \cos(ka + \sqrt{gk \sin\alpha}\, t) - \rho g \sin\alpha \\[2mm] \dfrac{\partial P}{\partial z} = -\rho g\ \cos\alpha. \end{cases}$$

Passing to Lagrangian coordinates, we obtain the system

$$\begin{cases} \dfrac{\partial P}{\partial a} = 0 \\[2mm] \dfrac{\partial P}{\partial b} = \rho g \sin\alpha\, \mathrm{e}^{2k(b-c)} - \rho g \sin\alpha \\[2mm] \dfrac{\partial P}{\partial c} = -\rho g \sin\alpha\, \mathrm{e}^{2k(b-c)} - \rho g \cos\alpha + \rho g \cos\alpha\, (1 + \tan\alpha)\, \mathrm{e}^{2kb_0}\, \mathrm{e}^{-2kc(1+\cot\alpha)} \end{cases}$$

with the solution

$$P = P_0 + \frac{\rho g \sin\alpha}{2k}\, \mathrm{e}^{2k(b-c)} - \rho g\, (c\ \cos\alpha + (b - b_0) \sin\alpha) - \frac{\rho g \sin\alpha}{2k}\, \mathrm{e}^{-2kc(1+\cot\alpha)}\, \mathrm{e}^{2kb_0}.$$

The obtained hydrodynamical pressure has the same value for any given particle as it moves about. At the free surface $c = (b_0 - b) \tan\alpha$ we have $P = P_0$ so that the dynamic boundary condition (i) is satisfied. The kinematic boundary condition at the free surface, (ii), holds as at any instance the free surface (4) is the image under (3) of the still water surface $\{c = (b_0 - b) \tan\alpha\ :\ b \leqslant b_0\}$. That there is no velocity normal to the sloping shore—this takes care of the boundary condition (ii)—is obvious, because at $z = 0$ we have $c = 0$ and the motion (3) is planar, without any velocity component in the direction of z. The proof that (3) is an explicit solution to the governing equations for water waves on a plane-sloping beach is complete.

3. Discussion

We constructed an exact edge-wave solution to the full water-wave problem, the graphical depiction of which is a fairly easy exercise. Let us now emphasize some of its significant properties. We present some simple observations which will provide a comprehensive description of this nonlinear wave and the particle motion it induces below the water surface.

The wavelength in the longshore direction $\lambda = 2\pi/k$ is related to the wave frequency ω by

$$\omega^2 = gk\,\sin\alpha$$

while the wave period is

$$T = \frac{2\pi}{\sqrt{gk\,\sin\alpha}}.$$

We easily infer that

$$\lambda = \frac{gT^2}{2\pi}\,\sin\alpha$$

so that the length of the edge wave is strongly dependent on its period and to a smaller degree on the beach slope. The phase velocity U of the edge wave (4) is given by

$$U = \sqrt{\frac{g\sin\alpha}{k}} \tag{5}$$

a fact consistent with the observation that if the bottom is flat ($\alpha = 0$) then $U = 0$ and no edge wave exists. The dispersion relation (5) for edge waves is obtained [14] within the confines of the formal linear approximation to the governing equations, but in our case the relation is derived rigorously as a byproduct of (3).

From (3) it is clear that any water particle describes circles as the edge wave passes—all these circles lie in planes parallel to the sloped bottom. The radius $\frac{1}{k}\,\mathrm{e}^{k(b-c)}$ of the circle described counterclockwise by a particle is maximal for the particles at the shoreline (that is, for $b = b_0,\ c = 0$).

As pointed out in the previous section, the motion of the water body induced by the passage of the edge wave (4) is rotational. The vorticity of the water flow defined by (3) is

$$\mathrm{curl}\,u = \left(-\frac{\partial u_2}{\partial z},\ \frac{\partial u_1}{\partial z},\ \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)$$

by the vanishing of u_3. Computing the inverse of the Jacobian matrix of the diffeomorphism (3) as

$$\begin{pmatrix} \frac{1+\exp[k(b-c)]\,\cos k(a+Ut)}{1-\exp[2k(b-c)]} & \frac{\exp[k(b-c)]\,\sin k(a+Ut)}{1-\exp[2k(b-c)]} & 0 \\ \frac{\exp[k(b-c)]\,\sin k(a+Ut)}{1-\exp[2k(b-c)]} & \frac{1-\exp[k(b-c)]\,\cos k(a+Ut)}{1-\exp[2k(b-c)]} & 0 \\ 0 & \frac{1}{(1+\tan\alpha)\,(1-\exp[2k(b_0-c-c\cot\alpha)])} & \frac{1}{(1+\tan\alpha)\,(1-\exp[2k(b_0-c-c\cot\alpha)])} \end{pmatrix}$$

a straightforward calculation yields the expression of the vorticity

$$\mathrm{curl}\,u = -\left(0,\ 0,\ \frac{2kU}{1-\,\mathrm{e}^{2k(b-c)}}\,\mathrm{e}^{2k(b-c)}\right)$$

for a particle whose parameters are $(a,\ b,\ c)$. Therefore, the vorticity is in the opposite sense to the revolution of the particles in their circular orbits, decreasing rapidly with distance from the shoreline/bed. Note that, despite the fact that the flow (3) is not two dimensional, the vorticity of each individual water particle is conserved as the particle moves about.

The run-up pattern is obtained by setting $z = 0$ in (4); this forces $b = b_0$ so that we have

$$\begin{cases} x = a - \dfrac{1}{k}\, \mathrm{e}^{kb_0} \sin\left(ka + \sqrt{gk\ \sin\alpha}\, t\right) \\ y = b_0 + \dfrac{1}{k}\, \mathrm{e}^{kb_0} \cos\left(ka + \sqrt{gk\ \sin\alpha}\, t\right) \\ z = 0 \end{cases} \tag{6}$$

with $a \in \mathbb{R}$. The above formula represents the parametrization of a smooth trochoid (if $b_0 < 0$) or of a cycloid with upward cusps (if $b_0 = 0$); it also explains why we imposed the condition $b_0 \leqslant 0$ as otherwise we would obtain a self-intersecting curve.

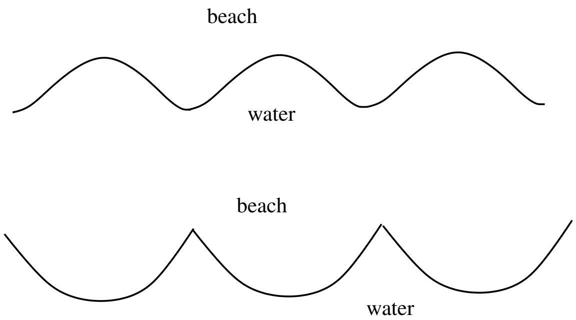

Figure 2. Run-up patterns at a fixed instance, viewed in the (x, y)-plane: trochoid (top) and cycloid (bottom).

A strong confirmation of a cusped run-up exists in the photograph of edge waves on the beach in Alum Bay, England [16] while several pictures in Komar [3] demonstrate the trochoidal run-up pattern for edge waves.

Another aspect of interest is the amplitude of the edge wave. To determine the elevation with respect to the reference half-plane

$$\left\{ z = -\frac{\tan\alpha}{2k}\, \mathrm{e}^{2kb_0} + (b_0 - y)\tan\alpha :\ y \leqslant b_0 \right\}$$

we compute the distance of a point $(x,\ y,\ z)$ lying on the free surface (4) to this plane,

$$d = z\cos\alpha + (y - b_0)\sin\alpha + \frac{\sin\alpha}{2k}\, \mathrm{e}^{2kb_0}$$

with the understanding that positive/negative values on the right-hand side mean that the point lies above/below the plane. Since (with $b \leqslant b_0$)

$$d = \frac{\sin\alpha}{2k} \left(\mathrm{e}^{2kb(1+\tan\alpha)-2kb_0\tan\alpha} + 2\, \mathrm{e}^{kb(1+\tan\alpha)-kb_0\tan\alpha} \cos(ka + \sqrt{gk\ \sin\alpha}\, t) \right) \tag{7}$$

we see that the amplitude of the edge wave decays exponentially away from the shoreline (as $b \to -\infty$). The same conclusion is reached by a formal linear approximation [17] and explains why edge waves are called 'trapped waves'.

As expected and ensured by (7), the amplitude of the edge wave varies with the parameters $a \in \mathbb{R},\ b \leqslant b_0$. From (7) we also infer that, at a fixed $b \leqslant b_0$, the crests and troughs correspond to the maximal/minimal values of $\cos(ka + \sqrt{gk\ \sin\alpha}\, t)$. At a fixed time $t \geqslant 0$, we obtain the crest curves (with $m \in \mathbb{Z}$ fixed and $b \leqslant b_0$ playing the role of a running parameter)

$$\begin{cases} x = \dfrac{2m\pi}{k} - \dfrac{1}{k}\sqrt{gk\ \sin\alpha}\, t \\[2ex] y = b(1+\tan\alpha) - b_0\tan\alpha + \dfrac{1}{k}\,\mathrm{e}^{kb(1+\tan\alpha)-kb_0\tan\alpha} \\[2ex] z = (b_0 - b)(1+\tan\alpha)\tan\alpha - \dfrac{\tan\alpha}{2k}\,\mathrm{e}^{2kb_0}\big(1 - \mathrm{e}^{2k(b-b_0)(1+\tan\alpha)}\big) \end{cases}$$

and the trough curves

$$\begin{cases} x = \dfrac{(2m+1)\pi}{k} - \dfrac{1}{k}\sqrt{gk\ \sin\alpha}\, t \\[2ex] y = b(1+\tan\alpha) - b_0\tan\alpha - \dfrac{1}{k}\,\mathrm{e}^{kb(1+\tan\alpha)-kb_0\tan\alpha} \\[2ex] z = (b_0 - b)(1+\tan\alpha)\tan\alpha - \dfrac{\tan\alpha}{2k}\,\mathrm{e}^{2kb_0}\big(1 - \mathrm{e}^{2k(b-b_0)(1+\tan\alpha)}\big). \end{cases}$$

Note that for both the crest and trough curves the value of x (at a given time) is fixed: standing at that location and looking towards the sea, these curves, orthogonal to the shoreline, are fully visible at certain instants. Indeed, taking into account the fact that on a crest/trough curve the deviation from the reference plane is

$$\frac{\sin\alpha}{2k}\,\big(\mathrm{e}^{2kb(1+\tan\alpha)-2kb_0\tan\alpha} \pm 2\,\mathrm{e}^{kb(1+\tan\alpha)-kb_0\tan\alpha}\big) \qquad b \leqslant b_0$$

in view of (7), the monotonicity of the right-hand side shows that the deviation becomes smaller (in absolute value) with the distance from the shore. This feature can be recognized in the graphical representation of the edge wave given in figure 3.

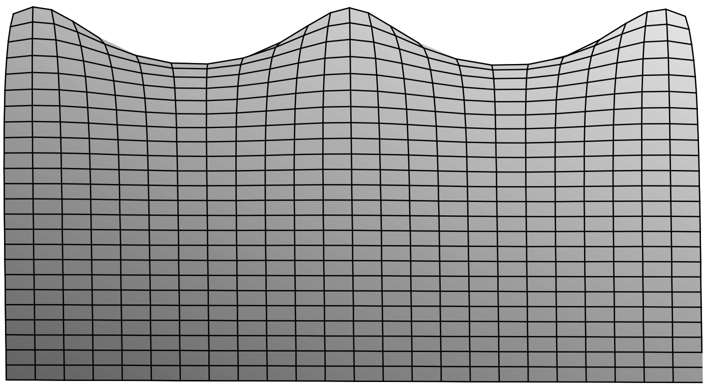

Figure 3. The edge wave viewed from offshore. The sinusoidal longshore structure and the exponential offshore decay in amplitude are visible.

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